Point-free geometries
Foundations and systems

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Outline

1. Point-based vs. point-free geometry
2. Half-plane structures
3. Oval structures

Point-based geometry

In Foundations of Geometry by K. Borsuk and W. Szmielew, with reference to David Hilbert's book of the same title, the authors examine structures of the form $\langle \mathbf{P}, \mathfrak{L}, \mathfrak{P}, \mathbf{B}, \mathbf{D} \rangle$, in which:

- $\mathbf{P}$ is a non-empty set of points,
- $\mathfrak{L}$ and $\mathfrak{P}$ are subsets of $\mathcal{P}(\mathbf{P})$,
- $\mathbf{B}$ and $\mathbf{D}$ are, respectively, ternary and quaternary relation in $\mathbf{P}$.
- Elements of $\mathfrak{L}$ and $\mathfrak{P}$ are called, respectively, lines and planes, $\mathbf{B}$ is called betweenness relation and $\mathbf{D}$ equidistance relation.
- We put specific axioms on $\mathbf{P}$, $\mathfrak{L}$, $\mathfrak{P}$, $\mathbf{B}$ and $\mathbf{D}$, and in this way we obtain a system of geometry that would probably satisfy Euclid and his contemporaries.

Incidence relation

- Sometimes an additional relation in $\mathbf{P} \times \mathfrak{L}$ and $\mathbf{P} \times \mathfrak{P}$ are introduced, the so called incidence relations, in our case will be denoted by 'ϵ'.
- In case p is a point and L is a line we read '$p \epsilon L$' as p is incident with L (similarly for planes)

Ontological commitments of region-based geometry

- Instead of the set of points we have the set of objects that are called solids, regions or spatial bodies. Let $\mathbf{R}$ be the set of all regions.
- $\mathbf{R}$ is ordered by the *part of* relation.
- The space $\mathbf{s}$ (if is assumed to exists) is usually the unity of $\mathbf{R}$
- Lines and planes are not elements of $\mathbf{R}$. Intuitively, $\mathbf{R}$ contains «regular» parts of space.

Points as distributive sets of regions

- Points are either sets of regions or sets of sets of regions. Let Π be the set of all points. Then:

$$\Pi \subseteq \mathcal{P}(\mathbf{R}) \qquad \text{or} \qquad \Pi \subseteq \mathcal{P}(\mathcal{P}(\mathbf{R}))\,.$$

- $\Pi \neq \mathbf{s}$ (the set of all points is not the space).

Figures as sets of points

- A figure is defined in a standard way, as a nonempty set of points:
$$\mathfrak{F} := \mathcal{P}_+(\Pi)\,.$$
- The set of all points is a figure: $\Pi \in \mathfrak{F}$.
- But:
$$\Pi \cap \mathbf{R} = \emptyset = \Pi \cap \mathfrak{F}\,,$$
that is points are neither regions nor abstract figures.
- Lines and planes, similarly as in classical geometry, are distributive sets of points: $\mathfrak{L} \cup \mathfrak{P} \subseteq \mathfrak{F}$.

From type-theoretical point of view

- In point-based geometries $\mathfrak{F}$ has the type $(*)$ in a hierarchy of types over the base set.
- In point-free approach it has either the type $((*))$ or $(((*)))$.

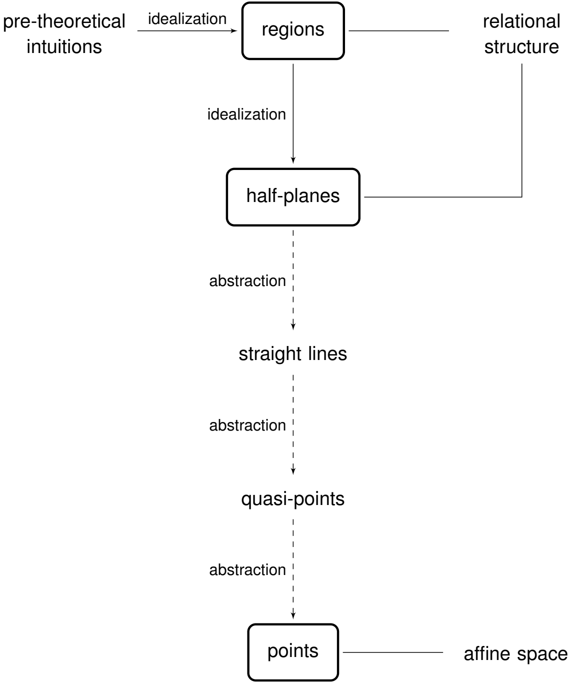
Summary

1. (i) $\mathbf{s} \neq \Pi$;
2. (ii) $\mathbf{s} \in \mathbf{R}$ and $\mathbf{s} \notin \mathfrak{F}$ (the space is one of regions and is not an «abstract» figure, that is it is not a distributive set of points);
3. (iii) $x \in \mathbf{R}$ and $x \neq \mathbf{s}$ iff $x \sqsubset \mathbf{s}$ (every region which is different from the space is its part and conversely, every part of the space is a region);
4. (iv) $\Pi \subseteq \mathcal{P}(\mathbf{R})$ or $\Pi \subseteq \mathcal{P}(\mathcal{P}(\mathbf{R}))$ and $\mathfrak{L}, \mathfrak{P} \subseteq \mathfrak{F}$ (all points are sets whose elements are regions or sets of regions; all lines and planes are abstract figures, but they are not parts of $\mathbf{s}$).

In light of the above remarks we can say that the conditions (iii)–(iv) are natural assumptions of region-based geometry.

Keywords and goals

1. A. N. Whitehead and ovate class of regions
2. open convex subsets of $\mathbb{R}^2$ — «the litmus paper»
3. Aleksander Śniatycki and half-planes
4. affine geometry
5. follow geometrical intuitions


pre-theoretical intuitions
idealization
regions
relational structure
idealization
half-planes
abstraction
straight lines
abstraction
quasi-points
abstraction
points
affine space

Affine geometry

- it is what remains of Euclidean geometry when the congruence relation is abandoned
- geometry of betweenness relation
- study of parallel lines
- Playfair's axiom

Basic notions

We examine triples $\langle \mathbf{R}, \leq, \mathbf{H} \rangle$ in which:

- $\mathbf{R}$ is a non-empty set whose elements are called regions,
- $\langle \mathbf{R}, \leq \rangle$ is a complete Boolean lattice,
- $\mathbf{H} \subseteq \mathbf{R}$ is a set whose elements are called half-planes (we assume that $\mathbf{1}$ and $\mathbf{0}$ are not half-planes).

Specific axioms for half-planes

$$h \in \mathbf{H} \longrightarrow -h \in \mathbf{H} \tag{H1}$$

Specific axioms for half-planes

$$\begin{aligned}\forall_{x_1,x_2,x_3\in\mathbf{R}}(\exists_{h\in\mathbf{H}}\forall_{i\in\{1,2,3\}}(x_i \bigcirc h \wedge x_i \bigcirc -h) \vee \\ \exists_{h_1,h_2,h_3\in\mathbf{H}}(x_1 \leq h_1 \wedge x_2 \leq h_2 \wedge x_3 \leq h_3 \wedge \\ x_1 + x_2 \perp h_2 \wedge x_1 + x_3 \perp h_2 \wedge x_2 + x_3 \perp h_1))\end{aligned} \tag{H2}$$

Lines and parallelity relation

Definition (of a line)

$L \in \mathcal{P}(\mathbf{H})$ is a line iff there is a half-plane h such that $L = \{h, -h\}$:

$$L \in \mathfrak{L} \stackrel{\mathrm{df}}{\longleftrightarrow} \exists_{h \in \mathbf{H}} L = \{h, -h\}\,. \qquad (\mathtt{df}\,\mathfrak{L})$$

Definition (of parallelity relation)

$L_1, L_2 \in \mathfrak{L}$ are parallel iff there are half-planes $h \in L_1$ and $h' \in L_2$ which are disjoint:

$$L_1 \parallel L_2 \stackrel{\mathrm{df}}{\longleftrightarrow} \exists_{h \in L_1} \exists_{h' \in L_2} h \perp h'\,. \qquad (\mathtt{df}\parallel)$$

In case L_1 and L_2 are not parallel we say they intersect and write: '$L_1 \nparallel L_2$'.

Specific axioms for half-planes

$$\forall_{h_1,h_2,h_3\in\mathbf{H}}(h_2 \leq h_1 \wedge h_3 \leq h_1 \longrightarrow h_2 \leq h_3 \vee h_3 \leq h_2) \qquad \text{(H3)}$$

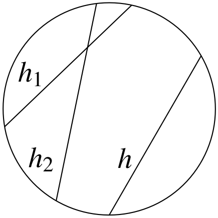


Figure: In Beltramy-Klein model there are half-planes contained in a given one but incomparable in terms of $\leq$. In the picture above h_1 and h_2 are both parts of h, yet neither $h_1 \leq h_2$ nor $h_2 \leq h_1$.

Angles and bowties. . .

Definition

- Given two intersecting lines L_1 and L_2 by an angle we understand a region x such that for $h_1 \in L_1$ and $h_2 \in L_2$ we have $x = h_1 \cdot h_2$:

$$x \text{ is an angle} \overset{\text{df}}{\longleftrightarrow} \exists_{L_1, L_2 \in \mathfrak{L}} \left(L_1 \nparallel L_2 \wedge \exists_{h_1 \in L_1} \exists_{h_2 \in L_2} \, x = h_1 \cdot h_2\right).$$

- An angle x is opposite to an angle y iff there are $h_1, h_2 \in \mathbf{H}$ such that $x = h_1 \cdot h_2$ and $y = -h_1 \cdot -h_2$.
- A bowtie is the sum of an angle and its opposite.

Notice that every pair $L_1 = \{h_1, -h_1\}$, $L_2 = \{h_2, -h_2\}$ of non-parallel lines determines exactly four pairwise disjoint angles: $h_1 \cdot h_2$, $h_1 \cdot -h_2$, $-h_1 \cdot h_2$ and $-h_1 \cdot -h_2$.

. . . and stripes

Definition

If $L_1 = \{h_1, -h_1\}$ and $L_2 = \{h_2, -h_2\}$ are parallel, yet distinct, lines and h_1 and h_2 are their disjoint sides, then $-h_1 \cdot -h_2$ is stripe.

Examples in the intended model

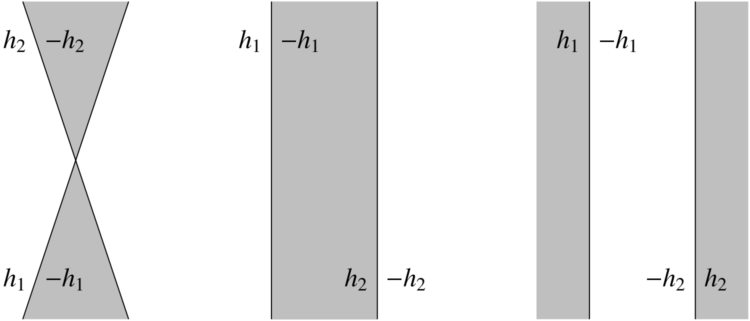


Figure: Fragments of a bowtie, a stripe and the complement of a stripe. These are all possible non-zero forms of the disjoint union of two distinct half-planes in the intended model. Any of the two shaded triangular areas of the bowtie is an angle.

Specific axioms for half-planes

$$h_1 \cdot h_2 \leq (h_3 \cdot h_4) + (-h_3 \cdot -h_4) \longrightarrow$$
$$h_3 = h_4 \vee h_1 \cdot h_2 \leq h_3 \cdot h_4 \vee h_1 \cdot h_2 \leq -h_3 \cdot -h_4 \,. \tag{H4}$$

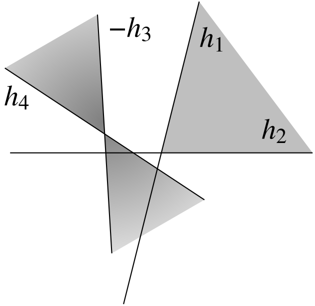


Figure: A geometrical interpretation of (H4).

Specific axioms for half-planes

$$h_1 \cdot h_2 \leq (h_3 \cdot h_4) + (-h_3 \cdot -h_4) \longrightarrow$$
$$h_3 = h_4 \vee h_1 \cdot h_2 \leq h_3 \cdot h_4 \vee h_1 \cdot h_2 \leq -h_3 \cdot -h_4 \,. \qquad \text{(H4)}$$

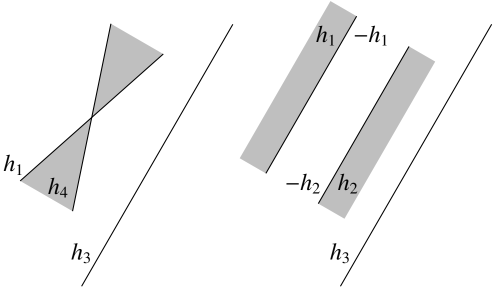


Figure: This two situations are excluded by the special case of (H4) in which $h_1 = h_2$.

Points

Definition

Given lines $L_1, \ldots, L_k$ by a net determined by them we understand the following set:

$$(L_1 \ldots L_k) \coloneqq \{g_1 \cdot \ldots \cdot g_k \mid \forall_{i \leqslant k}\, g_i \in L_i\}\,.$$

Lines $L_1, \ldots, L_k$ split a region x into m parts iff the set:

$$\{x \cdot a \neq \mathbf{0} \mid a \in (L_1 \ldots L_k)\}$$

has exactly m elements.

Points

Definition

- If $L_1, \ldots, L_k \in \mathfrak{L}$, an arbitrary element of the Cartesian product $L_1 \times \ldots \times L_k$ will be called an H-sequence.
- An H-sequence $\langle h_1, \ldots, h_k \rangle$ is positive iff $\{h_1, \ldots, h_k\}$ has a non-zero lower bound, otherwise it is non-positive.
- Two H-sequences $\langle h_1, \ldots, h_k \rangle$ and $\langle h_1^*, \ldots, h_k^* \rangle$ are opposite iff for all $i \leqslant n$, $h_i^* = -h_i$.
- Given a net $(L_1 \ldots L_k)$, regions $x, y \in (L_1 \ldots L_k)$ are opposite iff there are positive opposite H-sequences $\langle h_1, \ldots, h_k \rangle$ and $\langle h_1^*, \ldots, h_k^* \rangle$ in $L_1 \times \ldots \times L_k$ such that:

$$x = h_1 \cdot \ldots \cdot h_k \qquad \text{and} \qquad y = h_1^* \cdot \ldots \cdot h_k^* .$$

Points

Definition

A pseudopoint is any net (L_1L_2) such that $L_1 \times L_2$ contains four positive H-sequences.

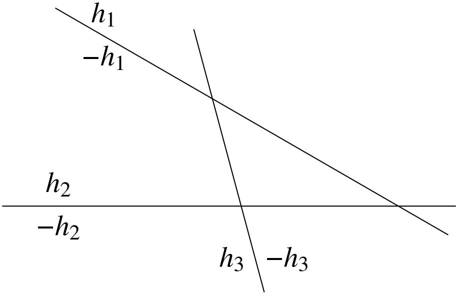
For any pseudopoint (L_1L_2), the lines L_1 and L_2 will be called its determinants. In case we have two pseudopoints (L_1L_2) and (L_1L_3) we say that they share a determinant L_1.

Points

Definition

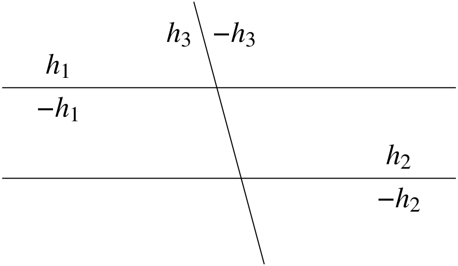
Lines L_1, L_2 and L_3 are tied iff $L_1 \times L_2 \times L_3$ contains two different non-positive and opposite H-sequences.

Non-tied lines



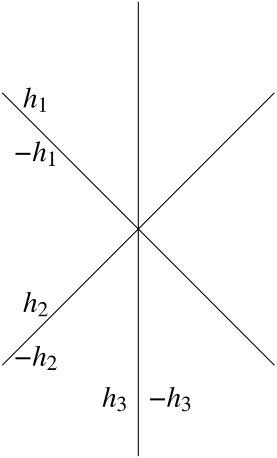
h_1	h_2	h_3	P
h_1	h_2	$-h_3$	P
h_1	$-h_2$	h_3	N
h_1	$-h_2$	$-h_3$	P
$-h_1$	h_2	h_3	P
$-h_1$	h_2	$-h_3$	P
$-h_1$	$-h_2$	h_3	P
$-h_1$	$-h_2$	$-h_3$	P

Non-tied lines



h_1	h_2	h_3	P
h_1	h_2	$-h_3$	P
h_1	$-h_2$	h_3	N
h_1	$-h_2$	$-h_3$	N
$-h_1$	h_2	h_3	P
$-h_1$	h_2	$-h_3$	P
$-h_1$	$-h_2$	h_3	P
$-h_1$	$-h_2$	$-h_3$	P

Tied lines

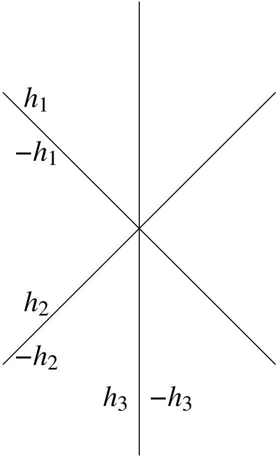


h_1	h_2	h_3	P
h_1	h_2	$-h_3$	P
h_1	$-h_2$	h_3	N
h_1	$-h_2$	$-h_3$	P
$-h_1$	h_2	h_3	P
$-h_1$	h_2	$-h_3$	N
$-h_1$	$-h_2$	h_3	P
$-h_1$	$-h_2$	$-h_3$	P

Points

Definition

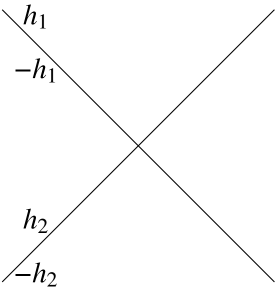
A pseudopoint $(L_1 L_2)$ lies on L_3 iff L_1, L_2 and L_3 are tied.



Points

Fact

$(L_1 L_2)$ lies on both L_1 and L_2.



h_1	h_2	h_1	P
h_1	h_2	$-h_1$	N
h_1	$-h_2$	h_1	P
h_1	$-h_2$	$-h_1$	P
$-h_1$	h_2	h_1	P
$-h_1$	h_2	$-h_1$	P
$-h_1$	$-h_2$	h_1	N
$-h_1$	$-h_2$	$-h_1$	P

Collocation

Definition

Psedopoints (L_1L_2) and (L_3L_4) are collocated (in symbols: $(L_1L_2) \sim (L_3L_4)$) iff (L_1L_2) lies on both L_3 and L_4.

Definition

Collocation of pseudopoints is an equivalence relation, therefore points can be defined as its equivalence classes:

$$\Pi := \pi/_{\sim} . \qquad (\mathtt{df}\,\Pi)$$

Incidence relation

Definition

$\alpha \in \Pi$ is incident with a line L iff there is a pseudopoint $(L_1 L_2) \in \alpha$ such that $(L_1 L_2)$ lies on L.

Betweenness relation

Definition

- $\alpha \in \Pi$ lies in the half-plane h iff there is $(L_1L_2) \in \alpha$ such that for every $x \in (L_1L_2)$, $x \cdot h \neq \mathbf{0}$.
- A line $L = \{h, -h\}$ lies between points α and β iff α lies in h and β lies in $-h$.

Definition

Points α, β and γ are co-linear iff some three pseudpoints from, respectively, α, β and γ share a determinant L.

Betweenness relation

Definition

A point γ is between points α and β iff there are $P \in \gamma$, $Q \in \alpha$ and $R \in \beta$ such that:

- P, Q and R share a determinant L (i.e. α, β and γ are co-linear) and
- a determinant L' of R which is different from L lies between α and β.

Śniatycki's Theorem

Theorem

Consider an H-structure:

$$\langle \mathbf{R}, \leq, \mathbf{H} \rangle .$$

Individual notions of point and line and relational notions of incidence and betweenness are definable in such a way that the corresponding structure $\langle \Pi, \mathfrak{L}, \epsilon, \mathbf{B} \rangle$ satisfies all axioms of a system of geometry of betweenness and incidence.

Basic notions

We will now consider structures $\langle \mathbf{R}, \leq, \mathbf{O} \rangle$ such that:

- elements of $\mathbf{R}$ are called regions,
- $\leq \subseteq \mathbf{R}^2$ is partial order,
- $\mathbf{O} \subseteq \mathbf{R}$ and its elements are called ovals.

First axioms

$$\langle \mathbf{R}, \leq \rangle \text{ is a complete atomless Boolean lattice.} \tag{O0}$$

$$\mathbf{O} \text{ is an algebraic closure system in } \langle \mathbf{R}, \leq \rangle \text{ containing } \mathbf{0}. \tag{O1}$$

$$\mathbf{O}^{+} \text{ is dense in } \langle \mathbf{R}, \leq \rangle. \tag{O2}$$

The hull operator

Definition

Let $\mathrm{hull}\colon \mathbf{R} \longrightarrow \mathbf{R}$ be the operation such that:

$$\mathrm{hull}(x) \coloneqq \bigwedge\{a \in \mathbf{O} \mid x \leq a\}\,. \tag{$\mathtt{df}\,\mathrm{hull}$}$$

For $x \in \mathbf{R}$ the object $\mathrm{hull}(x)$ will be called the oval generated by x.

Lines in the oval setting

Definition

By a line we understand a two element set $L = \{a, b\}$ of disjoint ovals, such that for any set of disjoint ovals $\{c, d\}$ with $a \leqslant c$ and $b \leqslant d$ it is the case that $a = c$ and $b = d$:

$$X \in \mathfrak{L} \overset{\text{df}}{\longleftrightarrow} \exists_{a,b \in \mathbf{O}^+} (a \perp b \wedge X = \{a, b\} \wedge \forall_{c,d \in \mathbf{O}^+} (c \perp d \wedge a \leqslant c \wedge b \leqslant d \longrightarrow a = c \wedge b = d)) . \qquad (\mathrm{df}\ \mathfrak{L})$$

For a line $L = \{a, b\}$ the elements of L will be called the sides of L.

Lines in the oval setting

Definition

Two lines $L_1 = \{a, b\}$ and $L_2 = \{c, d\}$ are paralell iff there is a side of L_1 which is disjoint from a side of L_2:

$$L_1 \parallel L_2 \stackrel{\text{df}}{\longleftrightarrow} \exists_{a \in L_1} \exists_{b \in L_2} \, a \perp b \,. \qquad (\mathtt{df}\parallel)$$

In case L_1 is not parallel to L_2 we say that L_1 and L_2 *intersect* and write '$L_1 \nparallel L_2$'.

Half-planes in the oval setting

Definition

A region x is a half-plane iff $x, -x \in \mathbf{O}^+$; the set of all half-planes will be denoted by '$\mathbf{H}$':

$$x \in \mathbf{H} \stackrel{\text{df}}{\longleftrightarrow} \{x, -x\} \subseteq \mathbf{O}^+ \ . \qquad (\mathtt{df}\,\mathbf{H})$$

Half-planes and lines in oval setting

Definition

Let $B_1, \ldots, B_n$ be non-empty spheres in $\mathbb{R}^2$ such that for $1 \leqslant i \neq j \leqslant n$: $\mathrm{Cl}\, B_i \cap \mathrm{Cl}\, B_j = \emptyset$. Consider the subspace $\mathscr{B}_n$ of $\mathbb{R}^2$ induced by $B_1 \cup \ldots \cup B_n$. Put:

- $\mathrm{r}\mathscr{B}_n := \{x \mid x$ is a regular open element of $\mathscr{B}_n\}$
- $\mathbf{O} := \{a \in \mathrm{r}\mathscr{B}_n \mid a = \bigcup_{1 \leqslant i \leqslant n} B_n \vee \exists_{1 \leqslant i \leqslant n} \exists_{b \in \mathrm{Conv}}\, a = B_i \cap b\}$

We will call $\mathbb{B}_n := \langle \mathrm{r}\mathscr{B}_n, \subseteq, \mathbf{O} \rangle$ the n-sphere structure.

Lines and half-planes in the oval setting

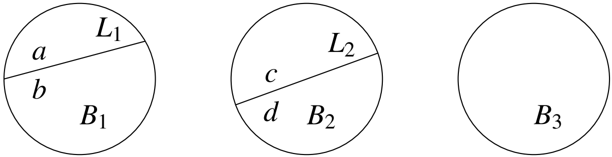


Figure: The structure $\mathbb{B}_3$.

Fact

For every $n \in \mathbb{N}$, $\mathbb{B}_n$ is a complete Boolean lattice and the axioms (O1) *and* (O2) *are satisfied in $\mathbb{B}_n$.*

Lines and half-planes in the oval setting

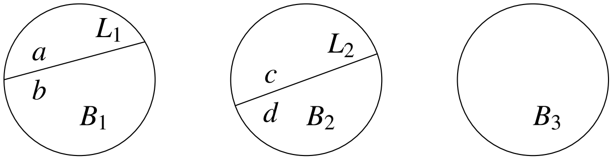


Figure: The structure $\mathbb{B}_3$.

Fact

For every $n \in \mathbb{N}$, the set of lines of $\mathbb{B}_n$ contains sets $\{B_i \cap h, B_i \cap -h\}$, where h is a half-plane in the prototypical structure $\mathbb{R}^2$ and both $B_i \cap h$ and $B_i \cap -h$ are non-empty. Two lines contained in different balls are always parallel.

Lines and half-planes in the oval setting

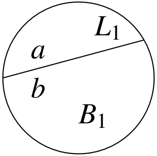


Figure: The structure $\mathbb{B}_1$.

Fact

In $\mathbb{B}_1$ the set of lines is equal to the set of all unordered pairs of the form $\{B_1 \cap h, B_1 \cap -h\}$. The sides of a line in $\mathbb{B}_1$ are half-planes in this structure.

Lines and half-planes in the oval setting

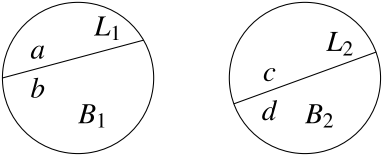


Figure: The structure $\mathbb{B}_2$.

Fact

B_1 and B_2 are the only half-planes of $\mathbb{B}_2$ and thus $\{B_1, B_2\}$ is the only line of $\mathbb{B}_2$ whose sides are half-planes. This line is parallel to every other line. In general, in $\mathbb{B}_n$ for $n \geqslant 2$ any pair $\{B_i, B_j\}$ with $i \neq j$ is a line parallel to every line in $\mathbb{B}_n$.

Lines and half-planes in the oval setting

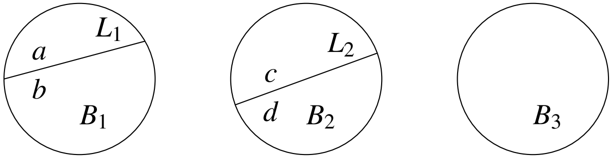


Figure: The structure $\mathbb{B}_3$.

Fact

There are no half-planes in $\mathbb{B}_n$ for $n \geqslant 3$, and thus there are no lines whose sides are half-planes.

Specific axioms

Definition

A finite partition of the universe $\mathbf{1}$ is a set $\{x_1, \ldots, x_n\} \subseteq \mathbf{R}$ whose elements are pairwise disjoint and such that $\bigvee\{x_1, \ldots, x_n\} = \mathbf{1}$. For a partition $P = \{x_1, \ldots, x_n\}$ and $x \in \mathbf{R}$ by *the partition of x induced by P* we understand the following set:

$$\{x \cdot x_i \mid 1 \leqslant i \leqslant n \wedge x \bigcirc x_i\} .$$

The sides of a line form a partition of $\mathbf{1}$; equivalently: the sides of a line are half-planes. (O3)

Specific axioms

For any $a, b, c \in \mathbf{O}$ which are not aligned there is a line which separates a from $\mathrm{hull}(b + c)$. (O4)

Specific axioms

If distinct lines L_1 and L_2 both cross an oval a, then they split a in at least three. (05)

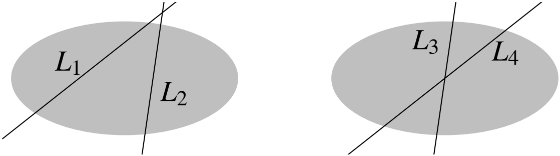


Figure: L_1 and L_2 split the oval into 3 parts, while L_3 and L_4 split it into 4 parts.

Specific axioms

No half-plane is part of any stripe and any angle. (06)

The purpose of (06) is to prove that parallelity of lines is transitive.

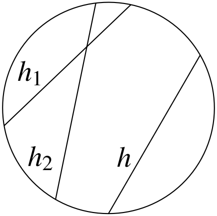


Figure: In Beltramy-Klein model: h is a part of the angle $h_2 \cdot -h_1$.

O-structures

Definition

A triple $\langle \mathbf{R}, \leq, \mathbf{O} \rangle$ is an O-structure iff $\langle \mathbf{R}, \leq, \mathbf{O} \rangle$ satisfies axioms (O0)–(O6).

Main theorems

Theorem

Let $\mathfrak{O} = \langle \mathbf{R}, \leqslant, \mathbf{O} \rangle$ be an O-structure and $\mathfrak{O}' := \langle \mathbf{R}, \leqslant, \mathbf{O}, \mathbf{H} \rangle$ be the structure obtained from $\mathfrak{O}$ by defining $\mathbf{H}$ as the set of all ovals whose complements are ovals. Then $\mathfrak{O}'$ satisfies all axioms for H-structures.

Theorem

If $\mathfrak{O}'$ is the extension of an O-structure $\mathfrak{O}$, then individual notions of point and line and relational notions of incidence and betweenness are definable from the operations and notions of $\mathfrak{O}'$ in such a way that all the axioms of a system of affine geometry are satisfied by the corresponding structure $\langle \mathbf{P}, \mathfrak{L}, \epsilon, \mathbf{B} \rangle$.

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